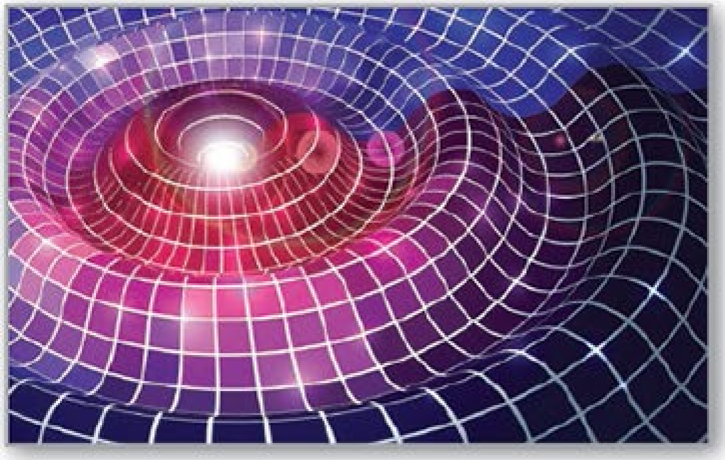
7 Eigenvalues And Eigenvectors




7.1 Eigenvalues and Eigenvectors

Objectives

- Verify eigenvalues and corresponding eigenvectors.
- Find eigenvalues and corresponding eigenspaces.
- Use the characteristic equation to find eigenvalues and eigenvectors, and find the eigenvalues and eigenvectors of a triangular matrix.
- Find the eigenvalues and eigenvectors of a linear transformation.

The Eigenvalue problem

The Eigenvalue problem (1 of 1)

Definitions of Eigenvalue and Eigenvector

Let A be an $n \times n$ matrix. The scalar λ is an **eigenvalue** of A when there is a *nonzero* vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$.

The vector $\mathbf{x}$ is an **eigenvector** of A corresponding to λ.

Example 1 – Verifying Eigenvectors and Eigenvalues

For the matrix

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

verify that $\mathbf{x}_1 = (1, 0)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = 2$, and that $\mathbf{x}_2 = (0, 1)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_2 = -1$.

Example 1 – Solution (1 of 2)

Multiplying $\mathbf{x}_1$ on the left by A produces

$$A\mathbf{x}_1 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \underbrace{2}_{\text{Eigenvalue}}\underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\text{Eigenvector}}.$$

So, $\mathbf{x}_1 = (1, 0)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = 2$.

Example 1 – Solution (2 of 2)

Similarly, multiplying $\mathbf{x}_2$ on the left by A produces

$$A\mathbf{x}_2 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -1\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

So, $\mathbf{x}_2 = (0, 1)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_2 = -1$.

Eigenspaces

Eigenspaces (1 of 1)

Theorem 7.1 Eigenvectors of λ Form a Subspace

If A *is an* $n \times n$ matrix with an eigenvalue λ, then the set of all eigenvectors of λ, together with the zero vector

$$\{\mathbf{x}: \mathbf{x} \text{ is an eigenvector of } \lambda\} \cup \{\mathbf{0}\}$$

is a subspace of R^n. This subspace is the **eigenspace** of λ.

Example 3 – Finding Eigenspaces in R^2 Geometrically

Find the eigenvalues and corresponding eigenspaces of

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

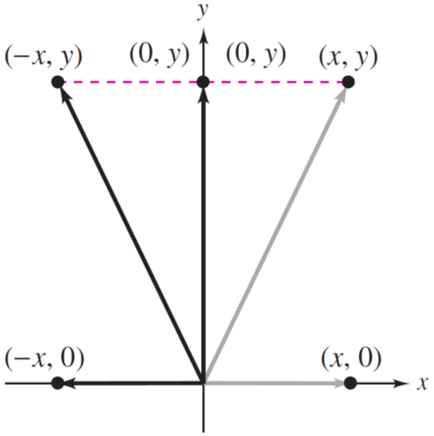
Solution:

Geometrically, multiplying a vector (x, y) in R^2 by the matrix A corresponds to a reflection in the y-axis. That is, if $\mathbf{v}$ = (x, y), then

$$A\mathbf{v} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}.$$

Example 3 – Solution (1 of 2)

Figure 7.1 illustrates that the only vectors reflected onto scalar multiples of themselves are those lying on either the *x*-axis or the *y*-axis.



A reflects vectors in the *y*-axis.

Figure 7.1

Example 3 – Solution (2 of 2)

For a vector on the x-axis

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} -x \\ 0 \end{bmatrix} = -1\begin{bmatrix} x \\ 0 \end{bmatrix}$$

Eigenvalue is $\lambda_1 = -1$.

For a vector on the y-axis

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix} = 1\begin{bmatrix} 0 \\ y \end{bmatrix}$$

Eigenvalue is $\lambda_2 = 1$.

So, the eigenvectors corresponding to $\lambda_1 = -1$ are the nonzero vectors on the *x*-axis, and the eigenvectors corresponding to $\lambda_2 = 1$ are the nonzero vectors on the *y-axis.*

This implies that the eigenspace corresponding to $\lambda_1 = -1$ is the *x*-axis, and that the eigenspace corresponding to $\lambda_2 = 1$ is the *y*-axis.

Finding Eigenvalues And Eigenvectors

Finding Eigenvalues and Eigenvectors (1 of 3)

The equation $\det(\lambda I - A) = 0$ is the **characteristic equation** of A. Moreover, when expanded to polynomial form, the polynomial

$$|\lambda I - A| = \lambda^n + c_{n-1}\lambda^{n-1} + \cdot \cdot \cdot + c_2\lambda^2 + c_1\lambda + c_0$$

is the **characteristic polynomial** of A. So, the eigenvalues of an $n \times n$ matrix A correspond to the roots of the characteristic polynomial of A.

Finding Eigenvalues and Eigenvectors (2 of 3)

Finding Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix.

1. Form the characteristic equation $|\lambda I - A| = 0$. It will be a polynomial equation of degree n in the variable λ.

2. Find the real roots of the characteristic equation. These are the eigenvalues of A.

3. For each eigenvalue λ, find the eigenvectors corresponding to λ_i by solving the homogeneous system $(\lambda_i I - A)\mathbf{x} = \mathbf{0}$. This can require row reducing an $n \times n$ matrix. The reduced row-echelon form must have at least one row of zeros.

Example 5 – Finding Eigenvalues and Eigenvectors

Find the eigenvalues and corresponding eigenvectors of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

What is the dimension of the eigenspace of each eigenvalue?

Example 5 – Solution (1 of 2)

The characteristic polynomial of *A* is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix}$$
$$= (\lambda - 2)^3.$$

So, the characteristic equation is $(\lambda - 2)^3 = 0,$ and the only eigenvalue is $\lambda = 2$. To find the eigenvectors of $\lambda = 2$, solve the homogeneous linear system represented

by $(2I - A)\mathbf{x} = \mathbf{0}$.

Example 5 – Solution (2 of 2)

$$2I - A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This implies that $x_2 = 0$. Using the parameters $s = x_1$ and $t = x_3$, you can conclude that the eigenvectors of λ = 2 are of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ s \text{ and } t \text{ not both zero.}$$

λ = 2 has two linearly independent eigenvectors, so the dimension of its eigenspace is 2.

Finding Eigenvalues and Eigenvectors (3 of 3)

Theorem 7.3 Eigenvalues Of Triangular Matrices

If A is an $n \times n$ triangular matrix, then its eigenvalues are the entries on its main diagonal.

Example 7 – Finding eigenvalues of Triangular and Diagonal matrices

Find the eigenvalues of each matrix.

a. $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 3 & -3 \end{bmatrix}$

b. $A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$

Example 7 – Solution

a. Without using Theorem 7.3,

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & 0 & 0 \\ 1 & \lambda - 1 & 0 \\ -5 & -3 & \lambda + 3 \end{vmatrix} = (\lambda - 2)(\lambda - 1)(\lambda + 3).$$

So, the eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 1$, and $\lambda_3 = -3$, which are the main diagonal entries of *A.*

b. In this case, use Theorem 7.3 to conclude that the eigenvalues are the main diagonal entries $\lambda_1 = -1$, $\lambda_2 = 2$, $\lambda_3 = 0$, $\lambda_4 = -4$, and $\lambda_5 = 3$.

Eigenvalues and Eigenvectors of Linear Transformations

Eigenvalues and Eigenvectors of Linear Transformations (1 of 1)

Eigenvalues and eigenvectors can also be defined in terms of linear transformations. A number λ is an **eigenvalue** of a linear transformation $T: V \to V$ when there is a nonzero vector **x** such that *T*(**x**) = λ**x**.

The vector **x** is an **eigenvector** of *T* corresponding to λ, and the set of all eigenvectors of λ (with the zero vector) is the **eigenspace** of λ.

Example 8 – Finding Eigenvalues and Eigenspaces

Find the eigenvalues and a basis for each corresponding eigenspace of

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Example 8 – Solution

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -3 & 0 \\ -3 & \lambda - 1 & 0 \\ 0 & 0 & \lambda + 2 \end{vmatrix}$$

$$= [(\lambda - 1)^2 - 9](\lambda + 2)$$

$$= (\lambda - 4)(\lambda + 2)^2$$

so the eigenvalues of A are $\lambda_1 = 4$ and $\lambda_2 = -2$. Bases for the eigenspaces are $B_1 = \{(1, 1, 0)\}$ and

$B_2 = \{(1, -1, 0), (0, 0, 1)\}$, respectively (verify these).