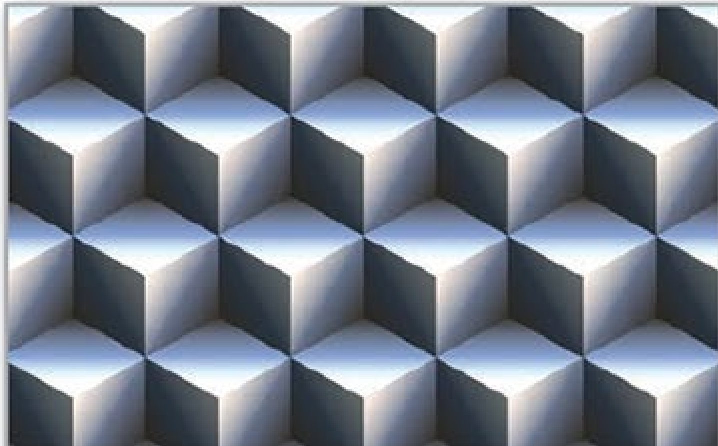
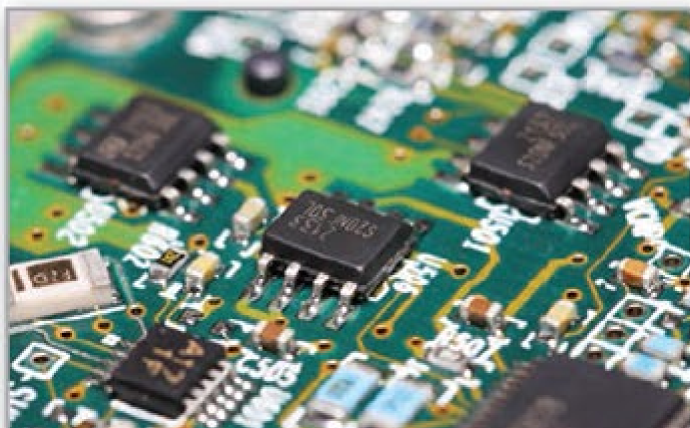
6 Linear Transformations





6.2 The Kernel and Range of a Linear Transformation

Objectives

- Find the kernel of a linear transformation.
- Find a basis for the range, the rank, and the nullity of a linear transformation.
- Determine whether a linear transformation is one-to-one or onto.
- Determine whether vector spaces are isomorphic.

The Kernel of a Linear Transformation

The Kernel of a Linear Transformation (1 of 3)

Definition of Kernel of a Linear Transformation

Let T: $V \to W$ be a linear transformation.

Then the set of all vectors $\mathbf{v}$ in V that satisfy $T(\mathbf{v}) = \mathbf{0}$ is the **kernel** of T and is denoted by $\ker(T)$.

Example 3 – Finding the Kernel of a Linear Transformation

Find the kernel of the projection T: $R^3 \rightarrow R^3$ represented by $T(x, y, z) = (x, y, 0)$.

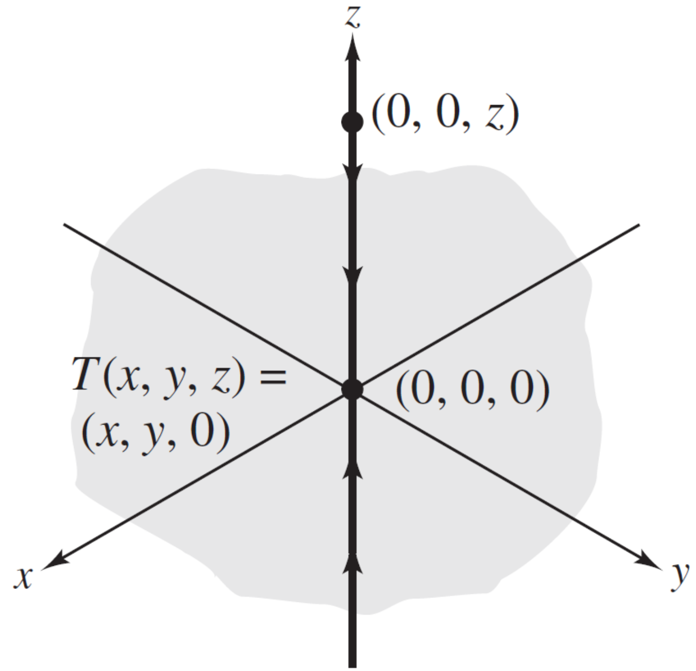
Solution:

This linear transformation projects the vector (x, y, z) in R^3 to the vector $(x, y, 0)$ in the xy-plane.

The kernel consists of all vectors lying on the z-axis.

Example 3 – Solution

That is, ker(T) = {(0, 0, z): z is a real number}. See Figure 6.1.



The kernel of T is the set of all vectors on the z-axis.

Figure 6.1

The Kernel of a Linear Transformation (2 of 3)

Theorem 6.3 The Kernel is a subspace of V

The kernel of a linear transformation $T: V \rightarrow W$ is a subspace of the domain V.

Example 6 – Finding a Basis for the Kernel

Define $T: R^5 \rightarrow R^4$ by $T(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x}$ is in R^5 and

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix}.$$

Find a basis for ker(T) as a subspace of R^5.

Example 6 – Solution (1 of 2)

Write the augmented matrix [*A* **0**] in reduced row-echelon form as shown below.

$$\begin{bmatrix} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{aligned} x_1 &= -2x_3 + x_5 \\ x_2 &= x_3 + 2x_5 \\ x_4 &= -4x_5 \end{aligned}$$

Example 6 – Solution (2 of 2)

Letting $x_3 = s$ and $x_5 = t$, you have

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s + t \\ s + 2t \\ s + 0t \\ 0s - 4t \\ 0s + t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \\ 0 \\ -4 \\ 1 \end{bmatrix}.$$

So a basis for the kernel of T is

$$B = \{(-2, 1, 1, 0, 0), (1, 2, 0, -4, 1)\}.$$

The Kernel of a Linear Transformation (3 of 3)

Theorem 6.3 Corollary

Let *T*: $R^n \rightarrow R^m$ be the linear transformation $T(\mathbf{x}) = A\mathbf{x}$.

Then the kernel of *T* is equal to the solution space of $A\mathbf{x} = \mathbf{0}$.

The Range of a Linear Transformation

The Range of a Linear Transformation (1 of 4)

The kernel is one of two critical subspaces associated with a linear transformation. The other is the range of T, denoted by range(T). The range of $T: V \to W$ is the set of all vectors $\mathbf{w}$ in W that are images of vectors in V. That is,

$$\text{range}(T) = \{T(\mathbf{v}) : \mathbf{v} \text{ is in } V\}.$$

Theorem 6.4 The Range of T Is a Subspace of W

The range of a linear transformation $T: V \to W$ is a subspace of W.

The Range of a Linear Transformation (2 of 4)

Theorem 6.4 Corollary

Let T: $R^n \rightarrow R^m$ be the linear transformation $T(\mathbf{x}) = A\mathbf{x}$.

Then the column space of A is equal to the range of T.

Example 7 – Finding a Basis for the range of a linear Transformation

For the linear transformation $R^5 \rightarrow R^4$ from previous example, find a basis for the range of *T*.

Solution:

Use the reduced row-echelon form of *A* from previous example.

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 7 – Solution

The leading 1's appear in columns 1, 2, and 4 of the reduced matrix on the right, so the corresponding column vectors of A form a basis for the column space of A.

A basis for the range of T is

$$B = \{(1, 2, -1, 0), (2, 1, 0, 0), (1, 1, 0, 2)\}.$$